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FINITE SAMPLE DISTRIBUTIONS OF RISK-RETURN RATIOS

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A B S T R A C T

This paper approximates the finite sample distributions of risk-return ratios using bootstrapped Gram-Charlier expansion, in the case of independent returns case. Under GARCH modeling for returns, we approximate risk-return ratios by bootstrap and bootstrapped Gram-Charlier expansion. Hansen method is also applied and a chi-squared approximation is proposed. We also apply our results for S&P 500 data set. Finally, a conclusion section is given.

Keywords: Bootstrap, Chi-squared approximation, Gram-Charlier expansion, Finite sample distribution, GARCH time series, Risk-return ratio, S&P 500.

INTRODUCTION

Portfolio construction is an important problem for investors. They should choose among risky asset in order to minimize the risk for a given target return. Indeed, a weight allocation problem is solved to obtain an optimum select. Markowitz's mean-variance method is a rational solution for quantitative finance experts. However, there are some other methods like the Capital Asset Pricing Model (CAPM) to price the risky asset in the presence of risk-less asset. Following Scherer and Martin (2005) (hereafter SM), a portfolio is optimum if there exists a relationship between marginal contribution of a specified return to portfolio risk and marginal implied return. The related slop is called Sharpe ratio. This index belongs to the big class of risk-return ratios denoted by ζ. Another important member of this family is Sortino ratio. In this paper, the finite distribution of these indices are studied

The above mentioned distributions have applications in calculation of Value at Risk (VaR). One of the main factors that exists in each financial activity is risk. This introduces uncertainty into financial problems and, therefore, decision making is made difficult under such conditions. Indeed, after famous financial disasters, it is advisable to re-estimate market risk. One risk measure is VaR (see, Kupiec, 1995). In many fields of applied statistics, estimators and test statistics are too complicated. Hansen (1992) refers the distribution of these statistics as non-standard distributions. It is clear that deriving the exact distribution of these statistics is too difficult. The computational statistics is a valuable tool to overcome to this difficulty. For a series of continuous-valued returns $r_i, i = 1, \dots, n$ these ratios are given by;

$$\zeta_{Sh} = \frac{\bar{r}}{S_r} \text{ and } \zeta_{So} = \frac{\bar{r}}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n} r_i^2 I(r_i < 0)}}.$$

where $\bar{r}$ and S_r are the sample mean and variance of $r_i, i = 1, \dots, n$. Notations ζ_{Sh} and ζ_{So} denote Sharpe and Sortino ratios, respectively. SM (2005) mentioned that the finite sample distributions of these ratios are too complicated and they suggested to use the re-sampling methods such as bootstrap.

As mentioned by SM (2005), the small sample properties of ζ_{Sh} and ζ_{So} are difficult to obtain. They approximated the distribution of ζ_{Sh} by a standard normal law. However, this select doesn't seem reasonable. Since suppose that r_i's come from a normal law with zero mean and variance σ^2. One can see that $\sqrt{n}\zeta_{Sh}$ has *t-student* distribution with $n-1$ degrees of freedom and note that the normal and t-student are not similar for small n's. As $n \to \infty$, $\sqrt{n}\zeta_{Sh}$ converges to a normal distributions. Therefore, the ζ_{Sh} may be approximated by a form of $c_n N(0, \sigma_n^2)$ for some positive sequences c_n. Note that the normal approximation is valid only for

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large sample sizes and SM don't specify the relation of estimated parameters to sample size. One can see that $\sqrt{n}\zeta_{So}$ converges to N(0, 2) distribution and $\sigma_n^{-2} r_1^2 I(r_i < 0)$ is distributed as mixture of two distributions which are degenerate law on 0 and chi-square distribution with one degree of freedom (in the normal case). Therefore, distribution of $\sigma_n^{-2} \sum_{i=1}^{n} r_i^2 I(r_i < 0)$ is too complicated.

However, for small and moderate sample sizes, it is much better to include more terms to approximate a specified distribution (say Gram-Charlier expansion). These terms are functions of moments of target distribution which we can estimate them by using the bootstrap technique (see Hall (1992)). An excellent reference about bootstrap is Efron and Tibshirani (1998). SM (2005) described the S+NUOPT module and related re-sampling co des.

SM (2005) assumed returns are independent observed random variables which is not true say for hedge fund data. They also suggested an autoregressive model for returns and bootstrapping $\zeta_{Sh}$and ζ_{So}. However, the usual model for returns is the GARCH time series. For example, Zivot and Wang (2003) modeled the daily Ford stock log returns using a GARCH(1,1) model.

This paper is organized as follows. Following Hall (1992), we use the combination of bootstrap and Gram-Charlier techniques to approximate distributions $\zeta_{Sh}$and ζ_{So}. We also bootstrap risk-return ratios under GARCH modeling for returns. It is seen again that it is better to bring our results in the form of series expansions. We also apply our results for S&P 500 time series. Finally, conclusions are given.

Gram-Charlier expansion: In this section, we study the series expansions for distributions of Sharpe and Sortino ratios. The Gram-Charlier series of the CDF of Sharpe and Sortino ratios are given by:

$$F(x) = N(z)\left\{1 + \frac{k_3}{3!\,\sigma^3} H_3(z) + \frac{k_4}{4!\,\sigma^4} H_4(z)\right\},$$

where $N(.)$ is the cumulative distribution function of Normal standard and $z = \frac{x-\mu}{\sigma}$. Hermit polynomials are $H_3(x) = x^3 - 3x$ and $H_4(x) = x^4 - 6x^2 + 3$. Here, μ, σ and k_r are the mean, standard deviation and r-th cumulants of specified ratio. These quantities are estimated by the bootstrap method.

Example 1. For $n = 200$ independent and identically distributed (iid) observations from standard normal distribution as Z_t, the $\theta = (\mu, \sigma, k_3, k_4)$ is (0, 0.005, 0, 0) for Sharpe ratio which again suggest a standard approximation and they are 5.4 × 10^{-4}, 1.04 × 10^{-2}, 3.3 × 10^{-4}, and 2.05 × 10^{-5}. Next, we change our positions to a distribution with heavier tail such as *t-student* with one degree of freedom. It is seen that the values of θ are the same as normal distribution for Sharpe ratio but they are too big for Sortino ratio. For example, the k_4 = 1120349269.

Example 2. For $n = 100$ iid observations from $N(\omega, v^2)$ distribution, the mean and variance of normal approximation of Sharpe ratio are given as follows. These quantities are estimated by applying a bootstrap method with 10000 repetitions.

Table 1. Normal parameters.

ω	v^2	mean	var
0	0.1	-0.0006	0.010772
0	1.2	0.0027	0.01115
1	0.1	301912	0.06161
1	1.2	0.9593	0.01398

GARCH modeling: It is known that the usual models for returns are the GARCH time series. Therefore, to find the θ in this case, it is enough to obtain the bootstrap re-samples $r_i^*, i = 1, \dots, n$. Let $r_i, i = 1, \dots, n$ be the GARCH(p, q) time series, that is,

$$r_i = \sigma_i Z_i$$

$$\sigma_i^2 = a_0 + \sum_{i=1}^{p} a_i r_{t-i}^2 + \sum_{j=1}^{q} b_j\, \sigma_{t-j}^2.$$

To derive r_i^*, it is enough to estimate a_0, $a_i, i = 1, \dots, p, b_j, j = 1, \dots, q$ and derive $\hat{\sigma}_i^2$ and $\hat{Z}_i$. Then, we suggest to re-sample $\hat{\sigma}_i^2$ and $\hat{Z}_i$ to derive r_i^*. Finally calculate $\zeta_{Sh}$and ζ_{So}. We repeat this procedure for $b = 1, 2, \dots, B$ (sufficiently large, say 10000).

Example 3. Consider a GARCH(1,1) model (see Bollerslev, 1986 and Vyazilov,1999) with $a_0 = 0.001$, $a_1 = 0.01$ and $b_1 = 0.95$. And generate 100 samples from this series. Next, suppose that the parameters are unknown and using the quasi-maximum likelihood method estimate these parameters where $\hat{a}_0 = 0.0098$, $\hat{a}_1 = 0.02$ and $\hat{b}_1 = 0.94$. We use the S+Finmetrics commands. Setting B = 10000 for n = 200, we plot the bootstrapped sampling distribution of $\zeta_{Sh}$and ζ_{So}in Figure 1. It is seen that the finite sample distribution of Sharpe ratio is approximated well by a normal distribution but this distribution however seems to be misleading for Sortino ratio.

However, using the bootstrap method, we can estimate θ as 3.3× 10^{-4}, 0.004, 7 × 10^{-6} and 1.19 × 10^{-6} for Sharpe ratio and they are 0.005, 0.082, 2.6 × 10^{-4} and 2.2 × 10^{-5}.

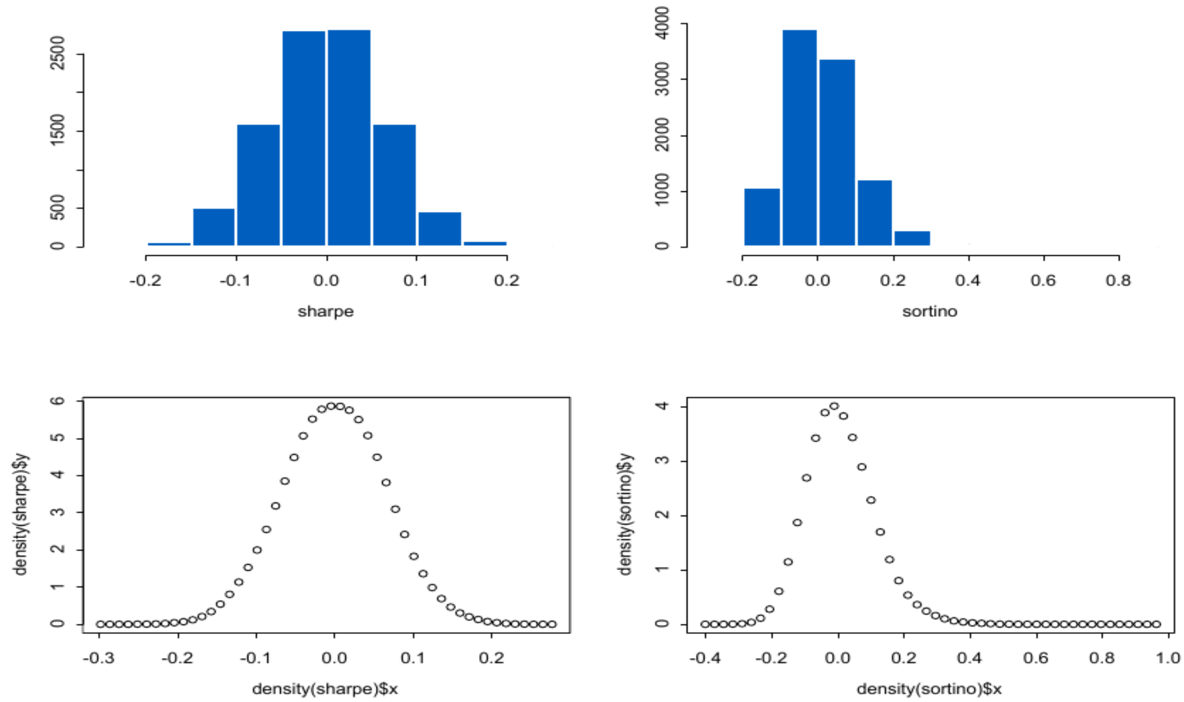

Figure 1. Histogram and density of bootstrapped rist-return ratios in GARCH (1,1), a0=0.001, a1=0.95, b1=0.01, n=250 and B=10000.

This shows that again the normal approximation is good for both Sharpe and Sortino ratios. The sight was changed to give more weight to arch parameter a_1. That is, let $a_0 = 0.001$, $a_1 = 0.95$ and $b_1 = 0.01$. Then, again the normal approximation works well for Sharpe ratio with θ = (0, 0.0041, 0, 0). However, these values are 0.00905, 0.0096, 7.38×10^{-4} and 1.8×10^{-4} . The empirical densities of risk-return ratios are given in Figure 2.

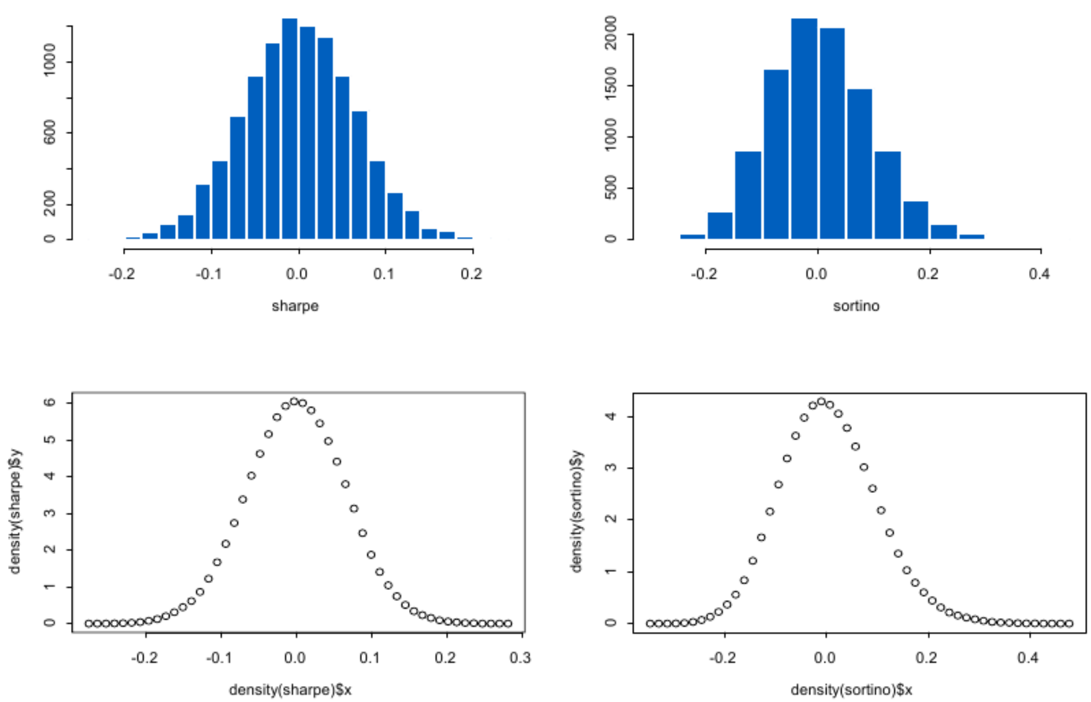

Figure 1. Histogram and density of bootstrapped rist-return ratios in GARCH (1,1), a0=0.001, a1=0.95, b1=0.01, n=250 and B=10000.

Example 4. The mean and variance of approximation normal distribution applied for the Sharpe ratio are 0 and 0.0039 for $a_0 = 0.001$, $a_1 = 0.01$ and $b_1 = 0.95$ and they are 0 and 0.00394 for $a_0 = 0.001$, $a_1 = 0.95$ and $b_1 = 0.01$.

Example 5. Here, an approximation, based on method of Hansen (1992) is proposed for the distribution of Sharp ratio. We understood that $m = 1$ is sufficient for all n and then:

$$\gamma_2(t|\theta) = \gamma(t) = \theta_0 + \theta_1 t.$$

Table 2 gives numerical approximations of for certain selected $\hat{\theta}_0$ and $\hat{\theta}_1$ for some selected sample sizes. Note that to the values of Table are $10\hat{\theta}_0$ and $10\hat{\theta}_1$. Error analysis is given in Table 2 including the maximum (max) and median (med) of 100 absolute errors. That is, the rows ”max” and ”med” report maximum and median absolute error across 100 fitting distributions for Sharp ratio. To keep the Table to a reasonable size, we report 1000×absolute errors. For example, the real maximum and median errors for $n = 9$ are $1.27/1000 = 0.00127$ and $0.22/1000 = 0.00022$, respectively. It is seen these errors are negligible.

Table 2 (a). Values of $\hat{v}$, $10\hat{\theta}_0$ and $10\hat{\theta}_1$ for Sharp ratio.

n	5	6	7	8	9	10	15	20	25
$\hat{v}$	1.04	1.11	1.2	1.27	1.29	1.32	1.44	1.54	1.55
$10\hat{\theta}_0$	0.585	0.5	0.68	0.814	0.612	0.609	0.541	0.617	0.477
$10\hat{\theta}_1$	8.265	6.926	6.001	5.524	4.627	4.143	2.815	2.064	1.631
n	30	35	40	50	60	70	80	90	100
$\hat{v}$	1.63	1.64	1.65	1.67	1.7	1.74	1.76	1.77	1.79
$10\hat{\theta}_0$	0.525	0.553	0.425	0.224	0.282	0.389	0.367	0.354	0.297
$10\hat{\theta}_1$	1.39	1.74	1.023	0.817	0.614	0.586	0.511	0.451	0.407

Table 2 (b). 1000×absolute Errors for Sharp ratio

n	5	6	7	8	9	10	15	20	25
Max	0.86	1.93	1.03	1.13	1.27	1.33	1.65	1.78	1.9
Med	0.57	0.101	0.25	0.58	0.22	0.13	0.44	0.52	0.66
n	30	35	40	50	60	70	80	90	100
Max	2.07	2.25	2.03	1.91	2.01	2.07	2.22	2.35	2.42
Med	0.16	0.14	0.7	0.61	0.36	0.55	0.13	0.48	0.15

Example 6. Here, we propose a chi-squared approximation in the form of $a_n\chi^2_{df_n}$for squared of Sortino ratio. The moment estimates of a_n and df_n are:

$$a_n = \frac{\pi_n^2}{2\mu_n} \text{ and } df_n = \frac{2\mu_n^2}{\pi_n^2}$$

where μ_n and π_n^2 are the mean and variance of squared of Sortino ratio. Table 3 gives the values of μ_n and π_n^2 and df_n. Table 4 gives the median and maximum of absolute error.

Table 3: Values of parameters.

n	10	20	30	40	60	80	100
μ_n	6.061	13.78	21.89	30.57	48.4	65.66	84.33
π_n^2	137.63	458.7	960.25	1711.4	3663.53	6216.98	1710.17
a_n	11.44	16.62	21.93	28	37.85	47.35	57.57
df_n	0.526	0.83	0.99	1.1	1.28	1.38	1.46

Table 4: Max and Median of errors.

n	10	20	30	40	60	80	100
Max	0.0015	0.0015	0.0014	0.0012	0.0016	0.0014	0.0015
Med	0.0005	0.0003	0.0004	0.0006	0.0004	0.0005	0.0005

Example 7. In this example, we let $a_1 = 0.02$ and $b_1 = 0.94$. We check the max and med of errors for some selected values for a_0's. The results are given in the Table 5.

Table 5. The max and med for various a_0.

a_0	0.01	0.008	0.005	0.0025	0.0017
max	0.056	0.088	0.12	0.78	1.5
med	0.023	0.045	0.09	0.53	0.87

Example 8. Zivot and Wang (2003) modeled the daily Ford stock returns using a GARCH (1,1) series with $a_0 = 6.53 \times 10^{-6}$, $a_1 = 0.074$, and $b_1 = 0.91$. We apply the aforementioned method here and we found that the max and med of $\frac{e_t}{t}$ for $t = 1, \ldots, 250$ are 0.032 and 0.0095, respectively.

Real data: The daily returns data on closing prices of the S&P 500 are accessible and valuable time series. The prices of this time series from 10/4/1983 to 8/30/1991, that is n = 2000 observations is considered. Here, the parameters of Gram-Charlier series θ is 0.04266, 0.0006815, 0 and 0 for Sharpe index and they are 0, 0.0082, 0 and 0 for Sortino ratio. This suggest that the bootstrapped normal approximation again works well in this case. The corresponding densities are given in Figure 3.

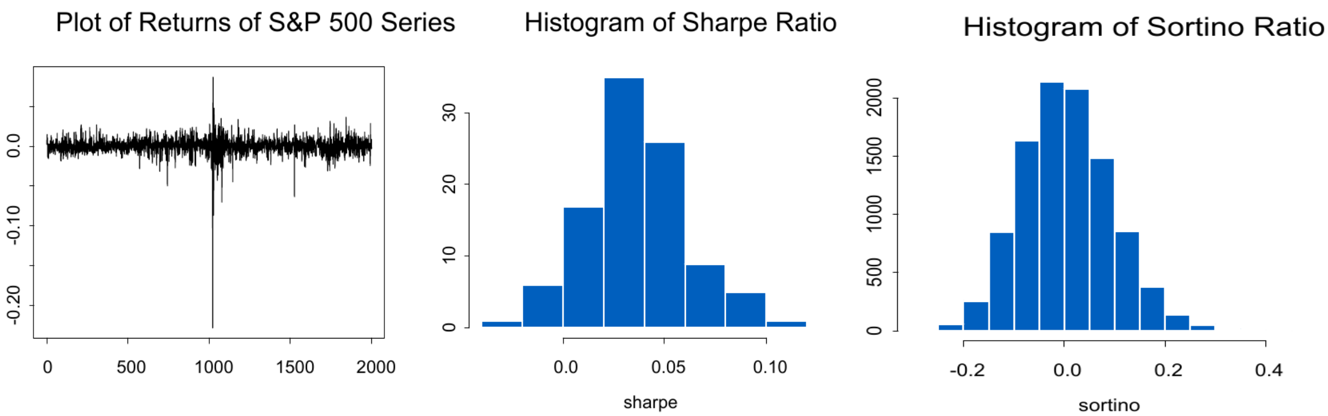

Figure 3. Time seires plot of S&P500 and histograms Sharp and Sortino indecies.

CONCLUSION

The finite sample distribution of risk-return ratios are approximated using Gram-Charlier expansion and bootstrap method under independent and GARCH modeling for returns. The S&P 500 are analyzed. Hansen method is also applied and a chi-squared approximation is proposed. We also apply our results for S&P 500 data set.

REFERENCES

Bollerslev, T. (1986). Generalized Autoregressive Conditional Heteroscedasticity. *Journal of Econometrics* 31, 307-327.

Efron, B. and Tibshirani, R. (1993). *An introduction to the bootstrap*. Chapman & Hall, New York.

Hall, P. (1992). *The bootstrap and Gram-Charlier expansion*. Springer.

Hansen, B. E. (1992). Testing for parameter instability in regressions with I(1) processes. *Journal of Business Economic Statistics* 10, 321-355.

Kupiec, P.H. (1995). Techniques for verifying the accuracy of risk measurement models. *The Journal of Derivatives* 3, 73-84.

Scherer, B. and Martin, D. (2005). *Introduction to portfolio optimization with NUOPT, S-PLUS, and S+Bayes*. Springer.

Vyazilov, A. E. (1999). The residual empirical distribution function in GARCH(1,1) and its application in hypothesis testing. *Russ. Math. Surv.* 54, 856-863.

Zivot, E. and Wang, J. (2003). *Modeling financial time series with S-plus*. Springer.